(1) Let k be a field and let $R = k[x, y, z, w]/(xy - zw)$. Construct a free resolution of the R-module $Rx + Rz$.

(2) Consider a commutative diagram of complexes with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_\bullet & \xrightarrow{\kappa} & L_\bullet & \xrightarrow{\lambda} & M_\bullet & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & K'_\bullet & \xrightarrow{\kappa'} & L'_\bullet & \xrightarrow{\lambda'} & M'_\bullet & \longrightarrow & 0 \end{array}$$

Show that if two out of α, β, γ are quasi-isomorphisms, then so is the third.

(3) Let R be a ring and let M and N be R-modules. An *extension* of M by N is a short exact sequence of R-modules

$$\mathcal{E}: \quad 0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$$

Two extensions $\mathcal{E}$ and $\mathcal{E}'$ of M by N are called *equivalent* if there is a homomorphism $\phi\colon E \to E'$ such that the diagram

$$\begin{array}{cccccccccc} \mathcal{E}: & 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\ & & & \| & & \downarrow{\scriptstyle\phi} & & \| & & \\ \mathcal{E}': & 0 & \longrightarrow & N & \longrightarrow & E' & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

commutes. Let $\mathrm{Ext}_R(M, N)$ denote the set of equivalence classes of extensions of M by N.

(a) Show that equivalence between extensions is indeed an equivalence relation.

(b) Show that $\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z})$ contains exactly n elements.